Petr Vopěnka

# New Infinitary Mathematics

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## Editor's Note

The original reason for this book was the consensus that Vopěnka's mathematical and philosophical contributions made after he left mainstream set theory should be available in English. Bringing the book to publication has taken ten years for the following reasons: first Vopěnka wrote another manuscript in Czech[1] subsequently translated by Hana Moravcová and Roland Andrew Letham, called *The Great Illusion of the Twentieth Century Mathematics*. However, it turned out that the translation of some parts of the text needed more relevant mathematical expertise and Alena Vencovská took on the task of making it correct. The author used the opportunity to extend and modify the book considerably. He worked on it until his sudden death in 2015. The result was twofold: more publications in Czech, namely the four-volumed work *New Infinitary Mathematics,*[2] along with *Prolegomena to the New Infinitary Mathematics,*[3] and a parallel English text with additions to the original book translated by Vencovská. The Czech and English versions differed little from each other, except that the order of the material was different, and Vopěnka left some parts out from the English version. In particular, he did not include what are now the first two chapters, and some sections throughout. This present version does include these initial chapters (on the theological foundations of Cantor's set theory and on its rise and growth, the former translated by Václav Paris) but it does not include all that is in the Czech version.

[1] Petr Vopěnka, *Velká iluze Matematiky XX. stoleti a nové základy* (Plzeň: Západočeská univerzita v Plzni a Nakladatelství Koniáš, 2011).

[2] Petr Vopěnka, *Nová infinitní matematika* (Praha: Karolinum, 2015).

[3] Petr Vopěnka, *Prolegomena k nové infinitní matematice* (Praha: Karolinum, 2013).

# Editor's Introduction

## About the Author

Petr Vopěnka grew up in the former Czechoslovakia, where he was born in 1935 (to parents who both taught mathematics at a secondary school). He enjoyed scouting in his youth and often remembered times spent at camps. In a way he remained true to the values he formed early on all through his life. Personal integrity, faith in truth prevailing over deceit, loyalty to friends, great love for his troubled country and an unshakeable commitment to his work were some of his most striking characteristics. To this, one needs to add that he loved to laugh.

For much of his life, Czechoslovakia was ruled by the communists: they took over in 1948, and education during Vopěnka's teenage years bore the stamp of Stalinism. Vopěnka reminisced about being asked to take turns in a whole day of reading funereal poems on the school radio upon Stalin's death in 1953, and he arrived in Prague later the same year to study mathematics in a city overlooked from a hill by Stalin's 16-meter-high statue. Fortunately, mathematics is relatively immune to ideological manipulation and Vopěnka remembered his student years and his teachers fondly.

His early research was mainly in topology and he wrote his master's thesis under the supervision of Eduard Čech, an eminent topologist and geometer, whose name lives for example in Čech cohomology and Čech-Stone compactification. Vopěnka used to say that Čech "showed him how to do mathematics". The research that he engaged in at that time concerned compact Hausdorff spaces and their dimensions.

Soon after graduation, Vopěnka started to teach mathematics at Charles University and he remained there for most of his professional life. Quite early on, he developed an interest in mathematical logic, championed in Czechoslovakia by Ladislav Rieger who wrote about the subject for the Czech mathematical community and ran a seminar on set theory. Vopěnka participated and, after Rieger's untimely death in 1962, took over as its organiser to provide strong and inspired leadership for Czechoslovak mathematical logicians. Vopěnka published work on nonstandard interpretations of Gödel-Bernays set theory based on using the ultrapower construction and then in collaboration with the seminar participants he contributed substantially to the exciting discoveries following Gödel and Cohen's groundbreaking work on the consistency and independence of the continuum hypothesis and the axiom of choice. Due to the Iron Curtain, communication with other mathematicians working in the area was limited and some results obtained independently in Prague came later than those in the West, but others remain credited to the Prague group. By all accounts it was as vibrant and fruitful a period as can be – Alfred Tarski wrote about the community in these words:[4]: "I do not know if there is at this point another place in the world, having as numerous and cooperative a group

[4] Quoted in Antonín Sochor, "Petr Vopěnka (born 16. 5. 1935)," *Ann. Pure Appl. Logic* 109 (2001): 1–8.

of young and talented researchers in the foundations of mathematics."

This lasted some years, but then two factors caused it to fall apart. One was that Vopěnka became very sceptical about the role that set theory, as it was, could have in truly explaining the phenomenon of infinity and in serving as a foundation for mathematics. It mattered to him; he did not wish to explore that intricate and bewitching maze any further so he started to look for alternatives. Paradoxically, the one concept which is today perhaps most strongly associated with Vopěnka within this area arose as he was abandoning the subject, when he proposed what became to be known as Vopěnka's principle. This yields a strong large-cardinal axiom that Vopěnka said he believed he could prove to be contradictory, suggesting it merely to make the point that investigating consequences of more and more set-theoretical axioms made little sense. However, Vopěnka's argument that it was contradictory contained an error, and interest in the axiom prospered outside of Czechoslovakia. Tightening controls within the country again limited communication with the West for academics like Vopěnka so it was some years later that he learnt with surprise that this principle was still alive and well established.

The other factor that contributed to the demise of this golden era of mainstream set theory in Prague were the political events – the 1960s brought a gradual thaw of orthodox communism leading to Prague Spring in 1968. This however was followed by the August 1968 invasion whereby the Warsaw Pact armies put an end to it. Some of Vopěnka's collaborators, in particular Tomáš Jech and Karel Hrbáček left the country, and most of the others sought their own independent paths. Vopěnka, who prior to 1968 had joined the efforts led by Alexander Dubček to reform communism and had gained some influence in running the Faculty of Mathematics and Physics at his university, would not support the official line after the invasion and might well have been forced to leave the university along with many other academics in similar positions. He was allowed to stay to do research, although his contact with students was very restricted. Many years later when he learnt that he owed this good fortune to the intervention of the Soviet mathematician P. S. Alexandrov, he used to joke that had he known how powerful a protector he had, he would have been braver (standing up to the suffocating pressure of the Czechoslovak communist "normalisation" of the 1970s and 1980s). In fact, he was one of the few who did stand up to it in any way that seemed possible.

At this turning point, Petr Vopěnka along with Petr Hájek wrote a book on semisets,[5] exploring set theories obtained by modifying the usual von Neumann-Bernays-Gödel axioms for classes and sets so that sets can have subclasses that are not themselves sets (proper semisets). Apart from the importance of semisets for forcing, Vopěnka's new motivation was investigating other ways in which the phenomenon of infinity could be captured mathematically, better reflecting how we encounter infinity when thinking about the world, often as a part of a large finite set. The book did not dwell on this aspect though and

[5] Petr Vopěnka and Petr Hájek, *The Theory of Semisets* (Prague: North Holland and Academia, 1972).

focused on providing a careful formal development of the theory of semisets and on showing its suitability for finding models of set theory via forcing.

Vopěnka then moved on to formulate a different set theory, which he hoped would capture his intuition about infinity in a better way. It was an intuition gained through much reflection on what we understand by infinity and how we see the world, influenced mainly by Bolzano and Cantor's writings, by discussions surrounding the birth of set theory and by the philosophy of Husserl and Heidegger (for many years there was a weekly seminar taking place in Vopěnka's office devoted to the study of their work). Semisets were a step in the right direction, but Vopěnka wished to formulate a new theory from the position of a mathematician free of any commitment to the current view of infinity; to develop mathematics as it might have been developed if satisfactory axioms for infinitesimals had been found before mathematics took its present course.

This led to what he called the alternative set theory. It contains sets and classes; sets alone behave as classical finite sets but they may contain subclasses which are not sets (semisets). Unlike Cantorian set theories, alternative set theory admits only two types of infinity: the countable infinite and the continuum. This is not a necessary requirement of such a set theory, it could be constructed otherwise, but Vopěnka's motivation was to keep only what could be justified by some intuition other than intuition arising purely from the study of set theory itself; for him it meant just the infinities associated with natural numbers or with the real line. A crucial principle in Vopěnka's alternative set theory is the Axiom of Prolongation, related to the phenomenon of the horizon (understood in a very general sense). It reflects the intuition that something seen to behave in a certain well-defined way as far as the horizon will continue to do so beyond the horizon.

Mathematically, the theory is close to the concept of nonstandard models of natural numbers underlying nonstandard analysis. However, from a foundational point of view there is a considerable difference since in nonstandard analysis infinitesimals are complicated infinitary objects whilst in AST some exist just as rational numbers do. Formulating a theory that allows mathematical analysis to be practiced in a way in which it was conceived by Leibniz, that is as a calculus with infinitesimals, was indeed one of Vopěnka's objectives. This had not been done within the alternative set theory at the time, and Vopěnka returned to the task in this book.

Vopěnka succeeded in assembling another group of enthusiastic mathematicians, who wanted to work with him and develop AST. One unfailingly supportive and faithful collaborator from before also joined him in the endeavour, Antonín Sochor. Interesting results were obtained, first within the Prague circle and later on also at other places in the world, but overall its impact was relatively small. In particular, investigations of alternative set-theoretical universes was restricted to what Vopěnka called a limit universe (as opposed to a witnessed universe). In a limit universe no "concrete" set such as the set of natural numbers less than $67^{293^{159}}$ can contain semisets but in a witnessed universe some can. The witnessed universes correspond to Vopěnka's intuition, but their theory is classically inconsistent (Vopěnka envisaged some approach involving the convincingness of proofs).

The first comprehensive account of AST appeared in 1979 in a monograph by Vopěnka.[6] In 1980 there was supposed to be a Logic Colloquium in Prague where AST would surely have been widely discussed and whatever stand logicians would have taken, its ambition to lead to new foundations for mathematics would have attracted more attention. However, shortly before the Colloquium was due to start the communist regime revoked the permission for it to take place, because the logic community was calling for the release of an imprisoned Czech logician and the regime feared the negative publicity. The next Logic Colloquium in Prague had to wait eighteen years, nine years after the Velvet Revolution. Vopěnka was an honorary chairman and his opening words are very telling, both of the man and the bygone times:

> "Ladies and Gentelmen, I am very happy to be able to welcome you to Prague. French historian Ernst Denis once wrote that in Prague every stone tells a story. As you walk across the Charles Bridge, pause to remember Tycho Brahe and Johannes Kepler who used to stroll there over 400 years ago as well as Bernard Bolzano two centuries later. I am sure that you too will fall in love with this old, inspiring, majestic, but also tragic city."
>
> These were the words with which I had planned to welcome participants of Logic Colloquium '80 which was cancelled by the communist government. The totalitarian regime was afraid that the participating mathematicians would call for the release of their colleague, mathematician Vaclav Benda, who was serving a five year prison term. He was imprisoned for publicly drawing attention to politically motivated prosecution of those opposing the regime. For us, Czech mathematicians, the cancellation meant even deeper isolation from our colleagues abroad. But we never doubted that even though mathematics is very beautiful, freedom is even more so. Logic Colloquium '98 will now commence.

By this time Vopěnka had entered yet another stage in his professional life. After the demise of communism in 1989 he had served as the Minister of Education in the new democratic government, throwing all his passion and energy into trying to reform the education system, with mixed success. After completing his term of office, he returned to academia but devoted himself mainly to the history and philosophy of mathematics. He wrote several books, in Czech, most notably *The Corner Stone of European Learning and Power* (Úhelný kámen evropské vzdělanosti a moci, 1998), *Narration about the Beauty of Neo-baroque Mathematics* (Vyprávění o kráse novobarokní matematiky, 2004) and *Meditations on the Foundation of Science* (Meditace o základech vědy, 2001). In 2004 he was awarded the Vize 97 prize by The Dagmar and Vaclav Havel Foundation designated by the charismatic Czech playwright president for "significant

[6] Petr Vopěnka, *Mathematics in the Alternative Set Theory* (Leipzig: Teubner, 1979).

thinkers whose work exceeds the traditional framework of scientific knowledge, contributes to the understanding of science as an integral part of general culture and is concerned with unconventional ways of asking fundamental questions about cognition, being and human existence." Vopěnka continued his work in the same spirit, eventually returning again to mathematics to describe his stand on its foundations.

### About the Book

There are some features in Vopěnka's work which it is useful to highlight. Vopěnka wrote extensively (in Czech) about ancient Greek geometry and its development throughout the centuries and about the origins and assumptions of set theory. It was essential for him to understand what mathematicians were doing, and he always wanted to see beyond the formal side of it: proving theorems from axioms did not suffice. He needed to know why anything should be assumed and this led him to formulate his own philosophical standpoint and develop his own terminology.

This is particularly important for his arguments about sets, which he discusses in this book. He explained his positions in detail for example in his book *Meditations on the Foundations of Science*.[7] It was influenced by the philosophy of Edmund Husserl and his followers but Vopěnka adapted the phenomenological program in his own way. The starting point are phenomena we encounter; from those we create objects by conceding them a "personality". It does not matter what is the character of the phenomenon in question, it could be something we perceive or remember or just think. When we single out some objects from those previously created, we can collect them together and when we consider them thus collected and without their various properties and interrelations, we make a collection of objects. Thus collections are determined exclusively by the presence of the objects belonging to them: belonging is not graded, and an object either belongs or not. When we consider a collection as an object, that is concede a personality to it, we make it into a class. The difference is that a collection is a multiplicity of objects but a class is a single object. As an object, it can belong to other collections. A class is uniquely determined by its members and, conversely, it uniquely determines the collection of its members although this can be in various ways and it may not be possible simply to list the members. A set is a class such that the collection of its members is sharply defined. For non-sharply defined collections, Vopěnka refers to examples like the numbers of grains taken from a heap of sand that still leave a heap. A semiset is defined to be a class which is a subclass of some set but not itself a set (where a class $X$ is a subclass of a class $Y$ if all members of $X$ belong to $Y$).

Apart from collections, Vopěnka uses the notion of domains. He writes

> When talking about people, we often think not only of people who are alive at that moment or have lived in the past, but also of those who are yet to be born or even of those who have never been born nor

[7] Petr Vopěnka, *Meditace o základech vědy* (Prague: Práh, 2001).

> ever will be. The extension of the concept of people is therefore not a collection, but a domain of all people. A domain is not a totality of existing objects (regardless of the modality of their existence); it is the source and simultaneously also a sort of container into which the suitable emerging or created objects fall. Naturally, every collection of objects can be interpreted as a domain, albeit an exhausted one. By actualising a domain we mean exhausting the domain, that is, substituting this domain by a collection of all the objects that fall or can fall into it.[8]

Thus it is some way from a domain to a set, and the questions of whether a domain can be actualised and whether this would yield a set is of fundamental importance.

In *Meditations*, Vopěnka gives an explanation of abstract objects and then he says:

> Abstract objects are the building blocks of the remarkable world of abstract mathematics. The modality of their being is some special, separated (abstract), and yet changeable being. These phenomena arise from nothingness by the strength of our will and their being culminates when they are captured in our minds. If we stop thinking them, they do not perish; just the modality of their being decreases. As if the nothingness slowly absorbed these phenomena but was no longer able to absorb them completely. Hence it at least hides them under the ever-condensing cover of emptiness from which they again surface when we remember them. We will refer to this idiosyncratic being of abstract objects as existence.[9]

It is in this light that we need to understand his arguments about Cantor's set theory and about the existence of the set of natural numbers. (By Cantor's set theory Vopěnka means any considerations based on Cantor's ideas, be it within the most commonly used ZFC – Zermelo Fraenkel set theory with the axiom of choice, or GB – Gödel Bernays set theory, within which he himself worked in the 1960s, or some other system based on the same approach to infinity.) For abstract objects with certain properties to exist, it must be possible to think them so at the very least there cannot be an apparent contradiction in them. But that is not all: we as finite beings should not really be able to think beyond the finite. So what is it that gives us the confidence to do so?

Vopěnka went further back, and started by asking how Euclidean geometry was possible. He argued that the mathematics of the ancient Greek world, that is, the way in which people thought about it, appropriated the capabilities of the Olympian gods to grasp the unchanging truth in the changing world. He refers to Zeus, or to a superhuman, as the performer of ancient (Greek) geometry. Zeus can extend a straight line further than any limit we may come up with

[8] Petr Vopěnka, *Prolegomena k nové infinitní matematice* (Prague: Karolinum, 2015).

[9] Petr Vopěnka, *Meditace o základech vědy* (Prague: Práh, 2001).

and he can see how a straight line approaches to touch a circle. Still, he does not wield absolute power and he could not hold in his mind all that there is. Such power does however belong to the God of medieval scholastic philosophy and using it made Cantor's set theory possible. It was in fact Bolzano, half a century before Cantor developed his theory of infinity, who came up with a proof of the existence of an infinite set (the only proof ever given, as Vopěnka used to say). Bolzano's proof is discussed in Section 1.3. Accordingly, Vopěnka sometimes refers to God, or to a God-man,[10] as the performer of the classical (modern) geometry and mathematics, on the grounds of it being based on Cantor's set theory. Faced with the question of how to perform mathematics now, Vopěnka notes that in the twenty first century, theological support is no longer there and he proposes his New Infinitary Mathematics, in the spirit of the alternative set theory.

The book has the following structure: Part I is a historical, philosophical and mathematical introduction. The author discusses the history of approaches to infinity up to the time when actually infinite sets became an integral part of mathematics. He shows how fundamental a role theological considerations played in enabling Bolzano and Cantor to produce work that established actually infinite sets as a legitimate object of study. Then he outlines the development of the basic ideas of set theory, focusing on the intuition that guided those early pioneers of set theory before the axiomatic frameworks found their final forms. He argues informally, attempting to capture the spirit of what appeared in the early days as the best way to build set theory; this includes the Axiom of Choice. Finally he argues that stripped of the support of medieval rational theology, we lose more than just certainty that actually infinite sets exist. To wit, assuming the actual existence of the set of all natural numbers (identified with their von Neumann's representations) leads, via the ultrapower construction and the ultraextension operator, to another set of all natural numbers containing all the previous ones and more, which is absurd. Although only some of the obvious questions and objections to this argument are answered in Vopěnka's text, one of his aims was to provoke a debate, and there is much that can be said. Part II proposes a new framework for mathematics while carefully motivating why it should be built in this way. The crucial concepts are those of natural real world, natural infinity and horizon. Mathematically, it is similar to the alternative set theory although there are differences, for example nothing corresponds to the axiom of two cardinalities which is adopted therein. Vopěnka saw his theory as an open challenge to be developed further; in particular he felt that predicate calculus may not be the only tool with which to study it. However, he did not investigate this further. In part III the author seeks to provide rigorous foundations for the development of the infinitesimal calculus on the basis of his theory. This is similar to Abraham Robinson's treatment of calculus in non-standard analysis, but Vopěnka's aim is to resurrect the original intuition that guided Leibniz, and to work with infinitesimals that actually exist as finite objects, without the need for them to be representatives of other, infinitary ob-

[10] See page 215.

jects. Part IV is shorter than the others and it is devoted to the real numbers. The four parts mirror the four volumes of the Czech version of *New Infinitary Mathematics*, with the first part including also some of *Prolegomena*. The circumstance that Vopěnka was simultaneously preparing this book and the Czech version[11] should explain some repetitions and variations in the present volume, although effort has been made to minimise them.

Regarding Vopěnka's style, it is useful to note that he frequently specifies the default meaning of symbols or letters at the start of various chapters or sections to be valid within those chapters or sections (or even during a section, to be valid until the end) and he does not necessarily repeat this when the symbols are used in theorems etc.

In the process of arranging for the publication of this work in English, some serious objections were raised, most notably the failure of the author to engage with the more recent scientific and philosophical literature and relate his thoughts to it. This is justified and could be damning, but there is much to redeem the book. It is a serious attempt by a leading mathematician to re-work the foundations of mathematics at a time when many mathematicians prefer to divorce their subject from the obligation to understand its own foundations. Cantorian set theory has in general been taken to provide such a foundation but the fact that there appears to be no one *true* classical set theoretical universe has made this hard to uphold. In a recent article[12] Akihiro Kanamori writes:

> Stepping back to gaze at modern set theory, the thrust of mathematical research should deflate various possible metaphysical appropriations with an onrush of new models, hypotheses, and results. Shedding much of its foundational burden, set theory has become an intriguing field of mathematics where formalized versions of truth and consistency have become matters for manipulation as in algebra. As a study couched in well-foundedness ZFC together with the spectrum of large cardinals serves as a court of adjudication, in terms of relative consistency, for mathematical statements that can be informatively contextualized in set theory by letting their variables range over the set-theoretic universe. Thus, set theory is more of an open-ended framework for mathematics rather than an elucidating foundation.

Still, some mathematicians and certainly philosophers of mathematics worry about the truth. Interesting as it would be, this book does not engage in a discussion of how it relates to such literature. Rather, it tries to find the truth from the position of a mathematician in the early 21st century, who spent a lifetime thinking about foundational issues, who is aware of the big metaphysical/theological assumptions behind the current framework and who searches for what is left when we give them up, relying just on human intuition and ability to make sense of the world.

[11] See page xi.

[12] Akihiro Kanamori, "Set Theory from Cantor to Cohen," in *Handbook of the Philosophy of Science; Philosophy of Mathematics,* ed. Andrew Irvine (Elsevier, 2007).